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AN INVESTIGATION OF PARTIAL ASYMPTOTIC STABILITY*

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The problem of the partial attraction of motion and the asymptotic stability of unperturbed motion is investigated, on the assumption that there exists a Lyapunov function with a positive or negative definite derivative. The solution of the problem is based on defining certain dynamical properties of the positive limit set, of the continuity and invariance type. The results, modify and generalize various well-known theorems of partial asymptotic stability. Examples are considered.

1. Consider the system of equations

$$x^{\cdot} = X(t, x); \quad X: R^+ \times \Gamma \to R^m \tag{1.1}$$
$$x \in R^m, \quad x = (y, z), \quad y \in R^s, \quad z \in R^p \ (m = s + p)$$
$$R^+ = [0, +\infty[, \ \Gamma = \{\|y\| \langle H \rangle 0, \ \|z\| < +\infty\},$$
$$\|x\| = \|y\| + \|z\|$$

($\|y\|$ is a norm in R^s and $\|z\|$ is a norm in R^p). The function X satisfies the conditions of the Carthéodory existence theorem /1/ and the conditions that ensure that the solutions are z-extendible /2/.

Let $x = x(t, t_0, x_0)$ be some solution of system (1.1) defined for all $t \geqslant t_0$. The partial positive limit set of this solution $\omega_y^+(x(t, t_0, x_0))$, is the set of points $y^* \in \Gamma_y = \{y \in R^s: \|y\| < H\}$, for each of which there exists a sequence $t_n \to +\infty$ such that $y(t_n, t_0, x_0) \to y^*$ /3/.

By imposing additional conditions on the right-hand side of (1.1), we can establish analytical properties of $\omega_y^+(x(t, t_0, x_0))$, of the continuity and invariance type.

Continuity property of $\omega_y^+(x(t, t_0, x_0))$. Let us assume that the function $Y(t, x)$: $R^+ \times \Gamma \to R^s$ satisfies the following condition: for every set $\Gamma_1 = \{\|y\| \leqslant H_1 < H, \ \|z\| < +\infty\}$ there exists a non-decreasing function μ_1: $R^+ \to R^+$ which is continuous at zero, $\mu_1(0) = 0$, and is such that for any continuous function u: $[a, b] \to \Gamma_1$

$$\left\| \int_a^b Y(\tau, u(\tau))\, d\tau \right\| \leqslant \mu_1(|b - a|) \tag{1.2}$$

If this condition is satisfied, then for every solution $x = x(t, t_0, x_0)$ of system (1.1) the function $\mu_1(t)$ is an estimate for the continuity of the y-component of the solution $y(t, t_0, x_0)$ for all $t \geqslant t_0$ such that $x(t, t_0, x_0) \in \Gamma_1$. In particular, if $x(t, t_0, x_0) \in \Gamma_1$ for all $t \geqslant t_0$, then $y(t, t_0, x_0)$ is continuous uniformly in $t \in [t_0, +\infty[$.

Hence it follows that, for some solution $x = x(t, t_0, x_0)$, if the set of y-limit points is such that $\omega_y^+(x(t, t_0, x_0)) \cap \Gamma_y \neq \varnothing$, then for every point $y^* \in \omega_y^+(x(t, t_0, x_0)) \cap \Gamma_y$ there exists a continuous function $y = \psi(t)$: $]\alpha, \beta[\to \Gamma_y$ such that $\psi(0) = y^*$ $(0 \in]\alpha, \beta[)$, and moreover $\{y = \psi(t): \alpha < t < \beta\} \subset \omega_y^+(x, (t, t_0, x_0))$.

The property of invariance of $\omega_y^+(x(t, t_0, x_0))$. Let us assume that the function $Y(t, x)$: $R^+ \times \Gamma \to R^s$ satisfies the following condition: for every set $\Gamma_1 = \{\|y\| \leqslant H_1, \ \|z\| < +\infty\}$ there exist two locally integrable functions $\lambda_1(t)$ and $\eta_1(t) \in L_1$ such that for all $t \in R^+$; $y, y_1, y_2 \in \Gamma_{1y} = \{y \in R^s: \|y\| \leqslant H_1\}$, $z \in R^p$,

$$\|Y(t, y, z)\| \leqslant \lambda_1(t) \tag{1.3}$$
$$\|Y(t, y_2, z) - Y(t, y_1, z)\| \leqslant \eta_1(t) \|y_2 - y_1\|$$

*Prikl. Matem. Mekhan., 55, 4, 539-547, 1991 and, moreover, $\lambda_1(t)$ is uniformly continuous in the mean and $\eta_1(t)$ is uniformly bounded in the mean, i.e.,

$$\int_E \lambda_1(\tau)\,d\tau \leqslant \varepsilon, \quad \int_t^{t+1} \eta_1(\tau)\,d\tau \leqslant N_1 \tag{1.4}$$

for any $\varepsilon > 0$, $t \in R^+$, any set $E \subset [t, t+1]$ of measure $m(E) \leqslant \delta_1(\varepsilon) > 0$ and some number N_1 /4/.

For every domain Γ_1 fix numbers δ_1 and N_1 as in (1.4). Define F_Ψ as the space of functions Ψ: $R \times \Gamma_y \to R^s$ for each of which and every domain Γ_{1y} there are two functions $\lambda_1(t, \Psi)$ and $\eta_1(t, \Psi)$ that satisfy inequalities of the form (1.4) with the fixed numbers δ_1 and N_1 and in addition, for all $t \in R$, $y, y_1, y_2 \in \Gamma_{1y}$,

$$\|\Psi(t, y)\| \leqslant \lambda_1(t, \Psi), \quad \|\Psi(t, y_2) - \Psi(t, y_1)\| \leqslant \eta_1(t, \Psi)\|y_2 - y_1\|$$

Using well-known results /4/, one can show that F_Ψ is a compact metrizable space.

For some number $H_0 < H$ we let $M_z(t, t_0) \subset \Gamma_z$ denote the set defined as the union over all $x_0 \in \Gamma_0 = \{\|x\| \leqslant H_0\}$ of the z-components of the solutions $x = x(t, t_0, x_0)$, i.e.,

$$M_z(t, t_0) = \bigcup \{z(t, t_0, x_0) : \|x_0\| \leqslant H_0\}$$

Let $z = z(t) \in M_z(t, t_0)$ be an arbitrary continuous function. Define $Y'(t, y) = Y(t, y, z(t))$. The family of shifts $\{Y_\tau'(t, y) = Y'(t+\tau, y), \tau \in R^+\}$, by the definition of F_Ψ, will be precompact in F_Ψ.

Consider some solution $x = x(t, t_0, x_0)$, $(t_0, x_0) \in R^+ \times \Gamma_0$ of system (1.1), defined for all $t \geqslant t_0$. The component $y(t) = y(t, t_0, x_0)$ is a solution of the first s equations of system (1.1), i.e., of the system

$$y^{\cdot} = Y'(t, y), \quad Y'(t, y) = Y(t, y, z(t)), \; z(t) = z(t, t_0, x_0)' \tag{1.5}$$

It can be shown /4/ that the precompactness of the family $\{Y_\tau'(t, y)\}$ and the existence of the limit functions Ψ implies the precompactness of system (1.5) and the existence of a family of limit systems

$$y^{\cdot} = \Psi(t, y), \quad \Psi \in F_\Psi \tag{1.6}$$

System (1.5) is regular in the sense that the solutions of system (1.6) have the uniqueness property.

The set $\omega_y^+(x(t, t_0, x_0))$ for a solution of system (1.1) to identical with the set $\omega^+(y(t))$ for the corresponding solution of system (1.1), which is quasi-invariant relative to the family of limit systems (1.6) /4/. Hence it follows that the set $\omega_y^+(x(t, t_0, x_0))$ is also quasi-invariant relative to system (1.6). To be precise: for every point $y_0 \in \omega_y^+ \cap \Gamma_y$ there exists a solution $y = \psi(t)$: $]\alpha, \beta[\to \Gamma_y$, $\psi(0) = y_0$ $(0 \in]\alpha, \beta[)$ of one of the limit systems (1.6) such that

$$\{y = \psi(t) : \alpha < t < \beta\} \subset \omega_y^+(x(t, t_0, x_0))$$

Remark 1. By analogy with (1.6), we can define a family of limit systems relative to the one-parameter family of functions $\{z_\nu(t) \in M_z(t, \nu), \nu \in R^+\}$. The right-hand side $\Psi(t, y)$ of the limit system is then defined as a limit point of a certain sequence $\{Y_\nu'(t, y) : \nu = \nu_n \to +\infty\}$.

Invariance properties of $\omega^+(x(t, t_0, x_0))$ and $\omega_y^+(x(t, t_0, x_0))$. Let us assume that the right-hand side of system (1.1) satisfies the following condition: for any compact subset $K \subset \Gamma$,

$$\|X(t, x)\| \leqslant \lambda_K(t), \quad \|X(t, x_2) - X(t, x_1)\| \leqslant \eta_K(t)\|x_2 - x_1\| \tag{1.7}$$

where the functions $\lambda_K, \eta_K \in L_1$ are such that there exist two numbers $N = N(K)$ and $\delta = \delta(K, \varepsilon) > 0$ that satisfy inequalities of type (1.4). Under this condition the family of shifts $\{X_\tau(t, x) = X(t+\tau, x), \tau \in R^+\}$ is precompact in some metrizable compact function space F_Φ /4/, system (1.1) may be associated with the family of limit systems

$$x^{\cdot} = \Phi(t, x), \quad \Phi \in F_\Phi \tag{1.8}$$

and moreover the complete positive limit set $\omega^+(x(t, t_0, x_0))$ is quasi-invariant with respect to (1.8). Thus, for solutions of system (1.1) which are bounded as functions of z, the set $\omega_y^+(x(t, t_0, x_0))$ is defined as the projection $(\omega^+(x(t, t_0, x_0)))_y$. If the solution is not bounded as a function of z, we proceed as follows.

Let us assume that the function $Z : R^+ \times \Gamma \to R^p$ satisfies the condition: for any continuous function $u = u(t) : R^+ \to \Gamma$ and every $\gamma \in [0, 1]$

$$\left\| \int_t^{t+\gamma} Z(\tau, u(\tau))\,d\tau \right\| \leqslant l(u) \tag{1.9}$$

When this is the case, the z-component $(z(t) = z(t, t_0, x_0))$ of a solution of (1.1) which is defined for all $t \geqslant t_0$ has bounded variation $\|z(t+\tau) - z(t)\| \leqslant l(T+1)$ over every interval $[0, T]$, uniformly in $\tau \in [t_0, +\infty[$. Consequently, if $\|z_k\| = \|z(t_k, t_0, x_0)\| \to +\infty$ as $t_k \to +\infty$, then for every $t \in R^+$ also $\|z(t_k + t, t_0, x_0)\| \to +\infty$ as $t_k \to +\infty$, and the sequence of functions $\|z(t_k + t, t_0, x_0)\|$ is bounded uniformly in $t \in [0, T]$.

Let us assume now that $Y(t, x)$ satisfies the following modification of conditions (1.3), (1.4): for every set $\Gamma_1 = \{\|y\| \leqslant H_1 < H, \|z\| < +\infty\}$, there exist a function $\lambda_1(t) \in L_1$ which is uniformly continuous in the mean (i.e., satisfies the first condition in (1.4)) and a constant N_1 such that for all $t \in R^+$, $(y, z) \in \Gamma_1$ and $y_1, y_2 \in \Gamma_{1y}$

$$\|Y(t, y, z)\| \leqslant \lambda_1(t), \tag{1.10}$$
$$\overline{\lim_{\|z\| \to +\infty}} \|Y(t, y_2, z) - Y(t, y_1, z)\| \leqslant N_1 \|y_2 - y_1\|$$

Analogous conditions will hold for every sequence of functions $Y_k'(t, y) = Y(t_k + t, y, z_k + z_k(t))$, where $t_k \to +\infty$ and $\{z_k: \|z_k\| \to +\infty\}$ are arbitrary sequences, $\{z_k(t)\}$ an arbitrary sequence of continuous functions which is uniformly bounded in $[0, T]$. Thus the sequence $\{Y_k'(t, y)\}$ turns out, as in the case (1.3), (1.4), to be precompact relative to a certain space F_Ψ of functions $\Psi: R \times \Gamma_y \to R^S$. Also, system (1.1), in addition to system (1.8), may be associated with a family of limit systems of the form (1.6) $y^\cdot = \Psi(t, y)$.

By dint of this construction, we have the following invariance property for the set $\omega_y^+(x(t, t_0, x_0))$. If $y(t_k, t_0, x_0) \to y^*$ as $t_k \to +\infty$ and the sequence $\{z_k = z(t_k, t_0, x_0)\}$ is bounded, there is a solution $x = \varphi(t) = (\psi(t), \theta(t))$ of the limit system such that $\psi(0) = y^*$, $y = \psi(t)$ is contained in $\omega^+_y(x(t, t_0, x_0))$ over the entire interval of definition $]\alpha, \beta[$ of the solution $x = \varphi(t)$. But if $\|z_k\| \to +\infty$, then there is a solution $y = \psi(t)$ of the limit system $y^\cdot = \Psi(t, y)$ such that $\psi(0) = y^*$, $\{\psi(t): \alpha < t < \beta\} \subset \omega_y^+(x(t, t_0, x_0))$, and the right-hand side $\Psi(t, y)$ of the system is a limit point of the sequence $\{Y_k'(t, y) = Y(t_k + t, y, z_k + z_k(t))$, $z_k(t) = z(t_k + t, t_0, x_0)\}$.

Remark 2. The additional restrictions imposed on $Z(t, x)$ make it possible to take the z-properties of system (1.1) into consideration as $\|z\| \to +\infty$. This formulation of the problem was considered in /5/.

2. Assume that there exists a continuous function $V(t, x)$: $R^+ \times \Gamma \to R$ for system (1.1), which satisfies a local Lipschitz condition with respect to x and thus has a derivative $V^{\cdot +}(t, x)$ /6/. Suppose that the derivative satisfies an inequality $V^{\cdot +}(t, x) \leqslant -W(t, x) \leqslant 0$, where W: $R^+ \times \Gamma \to R^+$ is some function satisfying the Carathéodory conditions, as in the case of $X(t, x)$.

Let us investigate the limiting behaviour of the solutions of system (1.1) as functions of y, depending on the conditions imposed on the right-hand side $X(t, x)$. To that end we need some definitions.

Let $t_n \to +\infty$ be a certain sequence and $t \in R$, $c \in R$ certain numbers. The set $P_\infty(t, c) \subset \Gamma_y$ is the set of points $y \in \Gamma_y$ for which there exist sequences $y_n \to y$ and $\{z_n \in R^p\}$ such that

$$\lim_{n \to \infty} V(t_n + t, y_n, z_n) = c$$

Let us assume that $W(t, x)$ satisfies conditions of type (1.3),

$$|W(t, x)| \leqslant \lambda_1(t), \quad |W(t, y_2, z) - W(t, y_1, z)| \leqslant \eta_1(t) \|y_2 - y_1\| \tag{2.1}$$

where $\lambda_1(t) \in L_1$ is uniformly continuous in the mean and $\eta_1(t) \in L_1$ is uniformly bounded in the mean, i.e., formulae similar to (1.4) are satisfied.

As done previously for $Y(t, x)$, it can be shown that there exists a compact metrizable space F_Ω of functions Ω: $R \times \Gamma_y \to R^+$ in which the family of shifts $\{W_\tau'(t, y) = W'(t + \tau, y)$, $W'(t, y) = W(t, y, z(t))\}$ is precompact for any continuous function $z(t) \in M_z(t, t_0)$. And for any sequence $t_n \to +\infty$ there exist a subsequence $t_{nj} \to +\infty$ and a function $\Omega \in F_\Omega$ such that, for any sequence of continuous functions $v_j(t)$: $[a, b] \to \Gamma_y$ which converges uniformly to $v^*(t)$: $[a, b] \to \Gamma_y$,

$$\int_a^b \Omega(\tau, v^*(\tau))\, d\tau = \lim_{j \to \infty} \int_a^b W(t_{nj} + \tau, v_j(\tau), z(t_{nj} + \tau))\, d\tau$$

We see that Ω is a limit function for W with respect to $z(t) \in M_z(t, t_0)$.

We shall view the set of values $\{y = \psi(t): \alpha < t < \beta\}$ as contained in $\{\Omega(t, y) = 0\}$ if, for any $t_1, t_2 \in]\alpha, \beta[$,

$$\int_{t_1}^{t_2} \Omega(\tau, \psi(\tau))\, d\tau = 0$$

Theorem 1. Assume that

1) $Y(t, x)$ satisfies condition (1.2);

2) there exists a function $V = V(t, x)$, bounded below on every set $R^+ \times \Gamma_1$, which has a derivative along trajectories of system (1.1) such that $V^{\cdot +}(t, x) \leqslant -W(t, x) \leqslant 0$, where W satisfies (2.1);

3) $x = x(t, t_0, x_0)$ is a solution of system (1.1) which is bounded as a function of $y, \|y(t, t_0, x_0)\| \leqslant H_1 < H$ for all $t \geqslant t_0$.

Then the set $\omega_y^+(x(t, t_0, x_0))$ of this solution is a union of subsets of continuous values $\{y = \psi(t)\colon -\infty < t < +\infty\} \subset \{P_\infty(t, c)\colon c = c_0 = \text{const}\} \cap \{\Omega(t, y) = 0\}$, where $\Omega(t, y)$ is the limit function for W with respect to $z = z(t, t_0, x_0)$ defined by the same sequence $t \to +\infty$ as $y = \psi(t)$.

Proof. It follows from conditions 2 and 3 of the theorem that there exists $c = c_0 = \text{const}$ such that

$$\lim_{t\to+\infty} V(t, y(t, t_0, x_0),\ z(t, t_0, x_0)) = c_0 \tag{2.2}$$

Suppose that $z(t) = z(t, t_0, x_0)$. Let $y^* \in \omega_y^+(x(t, t_0, x_0))$, in fact, let $y(t_k, t_0, x_0) \to y^*$ as $t_k \to +\infty$. By condition 1 of the theorem, there exist a subsequence $k_j \to \infty$ and a continuous function $y = \psi(t)\colon R \to \Gamma_y$ such that $\psi(0) = y^*$ and the sequence $y_{kj}(t) = y(t_{kj} + t, t_0, x_0)$ converges uniformly in $t \in [-T, T]$ to $y = \psi(t)$. Moreover, $\{y = \psi(t)\colon t \in R\} \subset \omega_y^+$, and by (2.2) we have

$$\lim_{j\to\infty} V(t_{kj} + t,\ y_{kj}(t),\ z(t_{kj} + t)) = c_0$$

Hence it follows that $\psi(t) \in \{P_\infty(t, c)\colon c = c_0 = \text{const}\}$ for all $t \in R$.

The condition $V^{\cdot +}(t, x) \leqslant -W(t, x) \leqslant 0$ implies that

$$V(t_{kj} + t) - V(t_{kj}) \leqslant -\int_0^t W'_{kj}(\tau, y_{kj}(\tau))\, d\tau \leqslant 0$$

$$W_{kj}'(t, y_{kj}(t)) = W(t_{kj} + t,\ y_{kj}(t),\ z(t_{kj} + t))$$

Consequently, choosing a subsequence $k_{ji} \to \infty$ for which $\{W'_{kji}(t, y)\}$ converges to some limit function $\Omega(t, y)$ and letting $k_{ji} \to \infty$, we find that

$$\{y = \psi(t)\colon t \in R\} \subset \{\Omega(t, y) = 0\}$$

Thus, for every point $y^* \in \omega_y^+$ there exists a continuous function $\psi(t)\colon R \to \Gamma_y$ such that $\psi(0) = y^*$, $\{\psi(t)\colon t \in R\} \subset \omega_y^+$, $\psi(t) \in \{P_\infty(t, c)\colon c = c_0 = \text{const}\} \cap \{\Omega(t, y) = 0\}$ for all $t \in R$. This completes the proof.

The quasi-invariance property of the positive limit set $\omega^+(x(t, t_0, x_0))$ or $\omega_y^+(x(t, t_0, x_0))$ enables us to establish a qualitative modification of the result.

Let $Y(t, x)$ and $W(t, x)$ satisfy conditions (1.3) and (2.1), respectively. The limit functions for Y and W, that is, Ψ and Ω, respectively, form a limit pair (Ψ, Ω) if they are defined relative to $z(t) \in M_z(t, t_0)$ for the same sequence $t_k \to +\infty$. Define the set $P_\infty(t, c)$ for the same sequence.

Let $N(c)$ denote the maximum subset of $\{P_\infty(t, c)\colon c = \text{const}\} \cap \{\Omega(t, y) = 0\}$ which is invariant with respect to the system $y^\cdot = \Psi(t, y)$, and $N_*(c)$ the union of the sets $N(c)$ over all limit pairs (Ψ, Ω) relative to the function $z(t) \in M_z(t, t_0)$.

Theorem 2. Under the assumptions of Theorem 1, assume in addition that $Y(t, x)$ satisfies conditions (1.3).

Then there exists $c = c_0 = \text{const}$ such that $\omega_y^+(x(t, t_0, x_0)) \subset N_*(c_0)$, i.e., $y(t, t_0, x_0) \to N_*(c_0)$ as $t \to +\infty$.

Let us assume that the functions $X(t, x)$ and $Y(t, x)$ satisfy conditions (1.7) and (1.10), respectively, and $W(t, x)$ satisfies both condition (2.1) and a condition similar to (1.10). We borrow the following notation from /7/. If (Φ, Λ) is a limit pair for (X, W), defined together with the set $V_\infty^{-1}(t, c)$ by some sequence $t_n \to +\infty$, then $E(c)$ is a maximum subset of $\{V_\infty^{-1}(t, c)\colon c = \text{const}\} \cap \{\Lambda(t, x) = 0\}$ invariant with respect to the system $x^\cdot = \Phi(t, x)$, $E^*(c)$ the union of the sets $E(c)$ over all limit pairs (Φ, Λ) and $(E^*)_y$ the projection of E^* on the hyperplane $z = 0$.

Theorem 3. Under the assumptions of Theorem 1, assume in addition that the functions $X(t, x)$, $Y(t, x)$, $W(t, x)$ satisfy conditions (1.7) and (1.10).

Then there exists $c = c_0 = \text{const}$ such that $\omega_y^+(x(t, t_0, x_0)) \subset (E^*)_y \cup N_*$, i.e., $y(t, t_0, x_0) \to (E^*(c_0))_y \cup N_*(c_0)$ as $t \to +\infty$.

Remark 3. The set $N^*(c_0)$ in the conclusion of Theorem 3 contains the y-limit points of the solution $x = x(t, t_0, x_0)$ defined by sequences $\{y(t_k, t_0, x_0) \to y, \|z(t_k, t_0, x_0)\| \to +\infty\}$, and $E^*(c_0)$ contains the y-limit points of $x = x(t, t_0, x_0)$ such that $\|z(t_k, t_0, x_0)\| \leqslant l$ for all $t_k \to +\infty$.

3. If we assume that $X(t, 0) \equiv 0$, system (1.1) has the trivial solution $x = 0$. Reasoning from the previous results, depending on our assumptions concerning X, we can derive sufficient conditions for the solution $x = 0$ to be asymptotically stable with respect to the variables y. The conditions are stated below as theorems.

Theorem 4. Assume that

1) there exists $V = V(t, x)$, $V(t, 0) = 0$, which is positive definite as a function of y, $V(t, x) \geqslant h(\|y\|) \geqslant 0$ /9/, and has a derivative along trajectories of system (1.1), $V^{\cdot+}(t, x) \leqslant -W(t, x) \leqslant 0$;

2) for every function $\Omega(t, y)$ which is a limit function for $W(t, y, z(t))$ relative to an arbitrary function $z = z(t) \in M_z(t, t_0)$,

$$\{P_\infty(t, c)\colon c = \text{const} \geqslant 0\} \cap \{\Omega(t, y) = 0\} = \{y = 0\}$$

Then the solution $x = 0$ of system (1.1) is asymptotically y-stable.

The assumptions of this theorem relative to $V^{\cdot}$ are weaker than those of Rumyantsev's theorem /8, 9/ or of analogues of Marachkov's theorem /10, 11/.

Example 1. Consider the linear system

$$y^{\cdot} = -\sin^2 ty + z_1 - z_2 e^t, \quad z_1^{\cdot} = z_2 e^t, \quad z_2^{\cdot} = y e^{-t} \tag{3.1}$$

The derivative of this system for the function $2V = y^2 + (z_1 - z_2 e^t)^2$ is $V^{\cdot} = -\sin^2 ty^2 \leqslant 0$. Since $(z_1 - z_2 e^t)^2 \leqslant 2V \leqslant 2V_0$ along solutions of the system, the right-hand side of the first equation in (3.1) satisfies condition (1.2) along every solution. The limit function for $W(t, y) = \sin^2 ty^2$ is $\Omega(t, y) = y^2 \sin^2(t + \alpha)$, $\alpha = \text{const}$, $0 \leqslant \alpha < \pi$. By Theorem 3, the solution $y = z_1 = z_2 = 0$ is asymptotically stable. This result was obtained for a similar system in /9, 12/ by estimating the second derivative of V.

Theorem 5. Assume that condition 1 of Theorem 4 holds, and also the following condition: for every function $z(t) \in M_z(t, t_0)$ there exists at least one limit function $\Omega(t, y)$ for $W(t, y, z(t))$ such that $\{P_\infty(t, c)\colon c = \text{const} > 0\} \cap \{\Omega(t, y) = 0\} = \varnothing$.

Then the solution $x = 0$ of system (1.1) is asymptotically y-stable uniformly in x_0.

Theorem 6. In addition to condition 1 of Theorem 4, assume that $V(t, x) \leqslant h_2(\|x\|)$, and also that for every limit function $\Omega(t, y)$ relative to an arbitrary family of functions $\{z_n(t) \in M_z(t, \nu_n), \nu_n \to +\infty$ as $n \to \infty\}$, we have $\{P_\infty(t, c)\colon c = \text{const} > 0\} \cap \{\Omega(t, y) = 0\} = \varnothing$.

Then the solution $x = 0$ of system (1.1) is uniformly asymptotically y-stable.

The assumptions of Theorems 4-6 are weaker than the corresponding assumptions of numerous earlier results /2, 8-10, 13/.

Let us assume that $Y(t, x)$ satisfies conditions (1.3), (1.4), so that $\omega_y^+(x(t, t_0, x_0))$ is quasi-invariant with respect to systems (1.6).

Theorem 7. Assume that

1) there exists a function $V = V(t, x)$, $V(t, 0) = 0$, $V(t, x) \geqslant h(\|y\|) \geqslant 0$, $V^{\cdot+}(t, x) \leqslant -W(t, x) \leqslant 0$;

2) for every limit pair (Ψ, Ω) relative to an arbitrary function $z(t) \in M_z(t, t_0)$, the maximum subset of the set $\{P_\infty(t, c)\colon c = \text{const} \geqslant 0\} \cap \{\Omega(t, y) = 0\}$ which is invariant with respect to the system $y^{\cdot} = \Psi(t, y)$ consists at most of the point $y = 0$.

Then the solution $x = 0$ of system (1.1) is asymptotically y-stable.

Theorem 8. Suppose that in addition to condition 1 of Theorem 7 the following condition is also satisfied: relative to every function $z(t) \in M_z(t, t_0)$, there exists at least one limit pair (Ψ, Ω) such that the set $\{P_\infty(t, c)\colon c = \text{const} > 0\} \cap \{\Omega(t, y) = 0\}$ contains no solutions of the system $y^{\cdot} = \Psi(t, y)$.

Then the solution $x = 0$ of system (1.1) is asymptotically y-stable uniformly in x_0.

Theorem 9. In addition to condition 1 of Theorem 7, assume that $V(t, x) \leqslant h_2(\|x\|)$, and also that the following condition is satisfied: for every limit pair (Ψ, Ω) relative to an arbitrary family of functions $\{z_k(t) \in M_z(t, \nu_k), \nu_k \to +\infty$ as $k \to \infty\}$, the set $\{P_\infty(t, c)\colon c = \text{const} > 0\} \cap \{\Omega(t, y) = 0\}$ contains no solutions of the system $y^{\cdot} = \Psi(t, y)$.

Then the solution $x = 0$ of system (1.1) is uniformly asymptotically y-stable.

Example 2. Consider the system

$$y_1^{\cdot} = -y_1 + p(t) y_2 - y_1 y_2^2 \sin^2 t (1 + \sin^2 (z_1 + z_2)) \tag{3.2}$$
$$y_2^{\cdot} = p(t) y_1 - y_2 + y_1^2 y_2 \sin^2 t (1 + \sin^2 (z_1 + z_2))$$
$$z_1^{\cdot} = f_1(t, y, z), \ z_2^{\cdot} = f_2(t, y, z)$$

where $p(t)$ is a continuous function, $0 \leqslant p(t) \leqslant 1$, and so $f_1(t, 0, 0) = f_2(t, 0, 0) = 0$.

Limiting equations for the first two equations are

$$y_1^{\cdot} = -y_1 + p^*(t) y_2 - y_1 y_2^2 \sin^2 (t + \alpha)(1 + q^*(t)) \tag{3.3}$$
$$y_2^{\cdot} = p^*(t) y_1 - y_2 - y_1^2 y_2 \sin^2 (t + \alpha)(1 + q^*(t))$$

where $p^*(t)$ and $q^*(t)$ are limit functions for $p(t)$ and $\sin^2(z_1(t) + z_2(t))$, and so $q^*(t) \geqslant 0$.

The derivative of the function $V = (y_1^2 + y_2^2)/2$ along trajectories of system (3.2) is $V^{\cdot} \leqslant -(y_1 - y_2)^2 \leqslant 0$. It can be shown that the set $\{y_1^2 + y_2^2 > 0\} \cap \{y_1 = y_2\}$ contains no solutions of the limit system (3.3). By theorem 9, the trivial solution of system (3.2) is uniformly asymptotically y-stable.

Similarly, starting from Theorem 3, we can deduce results relating to asymptotic y-stability, asymptotic y-stability uniform in x_0, and also y-instability when the right-hand side of system (1.1) satisfies conditions (1.7), (1.9) and (1.10). They will not be presented here; similar results, incidentally, were obtained by other techniques and in a different form in /14, 15/.

Under conditions (1.7), (1.9), and (1.10), we can also prove the following

Theorem 10. Assume that

1) there exists a function $V = V(t, x)$ such that

$$h_1(\|y\|) \leqslant V(t, x) \leqslant h_2(\|x\|), \ V^{\cdot +}(t, x) \leqslant -W(t, x) \leqslant 0$$

2) for any limit pair (Φ, Λ) of (X, W), the set $\{V_\infty^{-1}(t, c): c = \text{const} > 0\} \cap \{\Lambda(t, x) = 0\}$ contains no solutions of the system $y^{\cdot} = \Psi(t, y)$;

3) for any limit pair (Ψ, Ω) of (Y, W) relative to an arbitrary sequence of continuous functions $\{z_k(t)\}$, the set $\{P_\infty(t, c): c = \text{const} > 0\} \cap \{\Omega(t, y) = 0\}$ contains no solutions of the system $y^{\cdot} = \Psi(t, y)$.

Then the solution $x = 0$ of (1.1) is uniformly asymptotically y-stable.

We also note that similar techniques will yield results concerning partial instability.

Example 3. Consider an inhomogeneous sphere rolling and rotating without loss of contact over a rough horizontal plane Oxy which is oscillating vertically according to the law $z = z(t)$ about a fixed axis $O_2 z \parallel Oz$, where the OZ axis is directed vertically upward. Let us consider the case in which the centre of mass of the sphere is not its centre O_1, but the central ellipsoid of inertia is an ellipsoid of revolution and the axis of symmetry passes through the centre of the sphere. The configuration of the body is determined by the Cartesian coordinates x, y of the point of contact of the sphere with the surface, the Résal angles θ and ψ and the angle of revolution φ about the dynamical axis of symmetry $O_1\zeta$, which is parallel to the plane Oxz /16/.

Using the notation of /16/, we will consider the stability of the two families of equilibrium positions for the sphere in which

$$\theta = 0, \ \psi = 0; \ \theta - \pi, \ \psi = 0$$

and the centre of mass is on a single vertical below and above the point O_1, respectively.

Taking $V = T/(g + z_1^{\cdot\cdot}) - ml \cos\psi \cos\theta$, where T is the kinetic energy of the body, as a Lyapunov function, we apply Theorem 9, to find that under the action of viscous friction, on the assumption that

$$g + z_1^{\cdot\cdot}(t) \geqslant \varepsilon = \text{const} > 0, 2h_1(g + z_1^{\cdot\cdot}) + z_1^{(3)}(t) C \geqslant \varepsilon$$
$$2h(g + z_1^{\cdot\cdot}(t)) + z_1^{(3)}(mR^2 + 2mlR + A + ml^2) \geqslant \varepsilon \,|$$

the first family of equilibrium positions is uniformly asymptotically stable with respect to $x^{\cdot}$, $y^{\cdot}$, $\theta^{\cdot}$, $\varphi^{\cdot}$, $\psi^{\cdot}$, θ, ψ; the second is unstable.

Example 4. A question of practical interest is that of the stability of steady motions of a Cardan-suspended gyroscope and the effect of the parameters on its stability /17, 18/.

Using the formulation and formulae given in /18/, we will consider the case of a system in which, driven by certain forces, the outer frame rotates at a constant angular velocity Ω about a vertical axis, the heavy asymmetric rotor has a variable angular velocity $d\varphi/dt = \omega(t)$, and the axis of the inner frame is horizontal. We will single out those motions of the system in which the axis of the rotor points along the vertical, and the angle of revolution of the inner frame is accordingly $\theta = 0$.

We will take as the Lyapunov function

$$V = {}^1/_2\,(A\cos^2\varphi + B\sin^2\varphi)\,\vartheta^{\cdot 2}/k\,(t,\ \varphi,\ \vartheta) + \sin^2({}^1/_2\vartheta),\ k\,(t,\ \varphi,\ \vartheta) =$$
$$C\Omega\omega\,(t) + Mga_* + (C' - B_2 - B\cos^2\varphi - A\sin^2\varphi)\,\Omega^2\cos^2({}^1/_2\vartheta) -$$
$$(A - B)\,\Omega\omega\,(t)\sin\varphi\cos^2({}^1/_2\vartheta)\ (\inf\,(k\,(t,\ \varphi,\ 0),\ 0 \leqslant \varphi \leqslant 2\pi) \geqslant k_0 > 0)$$

After some computations we find that $V^{\cdot} \leqslant -k_0\vartheta^{\cdot 2} \leqslant 0$, if

$$\inf\,(k\,(t, \varphi, 0)\,(2h - (A - B)\,\omega\sin^2\varphi) + (A\cos^2\varphi + B\sin^2\varphi) \times$$
$$\left(\frac{\partial k}{\partial t}(t, \varphi, 0) + \frac{\partial k}{\partial \varphi}(t, \varphi, 0)\,\omega\right),\ 0 \leqslant \varphi \leqslant 2\pi \geqslant k_0 > 0$$

Setting up the limiting equations and applying Theorem 9, we conclude that under these conditions the corresponding motion of the object is uniformly asymptotically stable with respect to $\vartheta^{\cdot}$ and ϑ. Analysis of the conditions indicates that the asymmetry of the rotor affects stability in an important way; at large ω values the coefficient of viscous friction h should be fairly large.

We note that when $\omega = -Mga_*/C\Omega$ the system may also move with the rotor axis horizontal, $\vartheta = \pi/2$. Following the previous analysis, we find the following sufficient conditions for this motion to be asymptotically stable with respect to $\vartheta^{\cdot}$ and ϑ:

$$k_0 = (B + B_2 - C')\,\Omega^2 - (A - B)\,Mga_*/C > 0$$
$$(2h - (A - B)\,Mga_*/(C\Omega))\,k_0 - A\,(A - B)\,(|\,\Omega\,|\,\omega^2 + \Omega^2\,|\,\omega\,|) > 0$$

where we have assumed, to fix our ideas, that $A > B$.

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